Remarks on "Calculation of the quarkonium spectrum and m_b, m_c to order α_s^4"

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In a recent paper, we included two-loop, relativistic one-loop, and second-order relativistic tree level corrections, plus leading nonperturbative contributions, to obtain a calculation of the lower states in the heavy quarkonium spectrum correct up to, and including, $O(\alpha_s^4)$ and leading Λ^4/m^4 terms. The results were obtained with, in particular, the value of the two-loop static coefficient due to Peter; this has been recently challenged by Schröder. In our previous paper we used Peter's result; in the present one we now give results with Schröder's, as this is likely to be the correct one. The variation is slight as the value of b_1 is only one among the various $O(\alpha_s^4)$ contributions. With Schröder's expression we now have $m_b=5001^{+104}_{-66}$ MeV, $\overline{m}_b(\overline{m}_b^2)=4454^{+45}_{-29}$ MeV, $m_c=1866^{+215}_{-133}$ MeV, $\overline{m}_c(\overline{m}_c^2)=1542^{+163}_{-104}$ MeV. Moreover, $\Gamma(\Upsilon\to e^+e^-)=1.07\pm0.28$ keV(expt$=1.320\pm0.04$ keV) and the hyperfine splitting is predicted to be $M(\Upsilon)-M(\eta)=47^{+15}_{-13}$ MeV.

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I. INTRODUCTION

In a recent paper [1], hereafter denoted by PYI, we evaluated the heavy quarkonium spectrum to order α_s^4. The ingredients for the calculation were the one-loop corrections [2] to the static potential, and its first- and second-order contributions to the spectrum, the relativistic and mixed relativistic–one-loop [3] corrections, the two-loop corrections to the static potential, and the leading nonperturbative corrections [4]. The two-loop correction to the static potential used was that calculated by Peter [5], but the calculation of Peter has recently been challenged by Schröder [6]. In this last reference, the result of Peter's is checked for all pieces except one of the contributions to the C_A^2 coefficient, where an error in Peter's evaluation is pointed out. In the present paper we give the results of the calculation using the Schröder results. Note that the variation is *very* small; the reason is that the two-loop static potential is only one of the several contributions to the $O(\alpha_s^2)$ calculation. For example, using the value of the two-loop coefficient found by Peter, we found

$$m_b=5015^{+110}_{-70}\ \text{MeV},\quad m_c=1884^{+222}_{-133}\ \text{MeV}\quad (P) \tag{1.1a}$$

to which correspond the modified minimal subtraction scheme ($\overline{\text{MS}}$) masses[1]

$$\overline{m}_b(\overline{m}_b^2)=4467^{+49}_{-30}\ \text{MeV},$$

$$\overline{m}_c(\overline{m}_c^2)=1558^{+170}_{-104}\ \text{MeV}\quad (P).$$

With Schröder's result, one now has

$$m_b=5001^{+104}_{-66}\ \text{MeV},$$

$$\overline{m}_b(\overline{m}_b^2)=4454^{+45}_{-29}\ \text{MeV}\quad (S) \tag{1.1b}$$

and, for the c quark,

$$m_c=1866^{+215}_{-133}\ \text{MeV},$$

$$\overline{m}_c(\overline{m}_c^2)=1542^{+163}_{-104}\ \text{MeV}\quad (S). \tag{1.1c}$$

For the leptonic decay of the Υ and the hyperfine splitting, the results with Schröder's value of b_1 are

$$\Gamma(\Upsilon\to e^+e^-)=1.07\pm0.28\ \text{keV}\ \ (\text{expt}=1.320\pm0.04\ \text{keV})$$

and

$$M(\Upsilon)-M(\eta)=46.6^{+14.8}_{-12.7}\ \text{MeV}.$$

They are almost identical to those obtained before.

II. EFFECTIVE POTENTIAL

We follow the method of effective potentials and the renormalization scheme of Ref. [3]. The Hamiltonian for quarkonium is

$$H=H^{(0)}+H_1, \tag{2.1a}$$

where

[1]These values are slightly different from the ones displayed in PYI due to an error in the formula relating the pole and $\overline{\text{MS}}$ masses used in Titard and Ynduráin [3] and in PYI, as communicated to us by K. G. Chetyrkin. See Eq. (2.4) for the corrected formula.

$$H^{(0)}=2m+\frac{-1}{m}\Delta-\frac{C_F\tilde{\alpha}_s(\mu^2)}{r},$$

$$\tilde{\alpha}_s(\mu^2)=\alpha_s(\mu^2)\left\{1+\left(a_1+\frac{\gamma_E\beta_0}{2}\right)\right)\frac{\alpha_s(\mu^2)}{\pi}$$
$$+\left[\gamma_E\left(a_1\beta_0+\frac{\beta_1}{8}\right)\right.$$
$$\left.+\left(\frac{\pi^2}{12}+\gamma_E^2\right)\frac{\beta_0^2}{4}+b_1\right]\frac{\alpha_s^2}{\pi^2} \tag{2.1b}$$

and will be solved exactly. H_1 is

$$H_1=V_{\rm tree}+V_1^{(L)}+V_2^{(L)}+V^{(LL)}+V_{\rm s.rel}+V_{\rm spin}, \tag{2.1c}$$

where the explicit expression for H_1 can be found in PYI. The running coupling constant has to be taken to three loops. a_1 was calculated in Ref. [1] and has been checked by independent calculations; the only discrepancy lies in the value of the coefficient[2] b_1. According to Peter [5],

$$b_1=\frac{1}{16}\left\{\left[\frac{4343}{162}+6\pi^2-\frac{1}{4}\pi^4+\frac{22}{3}\zeta(3)\right]C_A^2-\left[\frac{1798}{81}\right.\right.$$
$$\left.+\frac{56}{3}\zeta(3)\right]C_AT_Fn_f-\left[\frac{55}{3}-16\zeta(3)\right]C_FT_Fn_f$$
$$\left.+\frac{400}{81}T_F^2n_f^2\right\}\simeq 24.30, \tag{2.2a}$$

while Schröder [6] gives

$$b_1=\frac{1}{16}\left\{\left[\frac{4343}{162}+4\pi^2-\frac{1}{4}\pi^4+\frac{22}{3}\zeta(3)\right]C_A^2-\left[\frac{1798}{81}\right.\right.$$
$$\left.+\frac{56}{3}\zeta(3)\right]C_AT_Fn_f-\left[\frac{55}{3}-16\zeta(3)\right]C_FT_Fn_f$$
$$\left.+\frac{400}{81}T_F^2n_f^2\right\}\simeq 13.2. \tag{2.2b}$$

Although the difference between the two lies only in one piece of the coefficient of $\frac{1}{16}C_F^2$, a $4\pi^2$ vs a $6\pi^2$, the Schröder result makes the two loop correction much smaller (and hence the overall calculation more believable). We present here results with Eq. (2.2b) since they are the ones more likely to be correct [moreover, results with Eq. (2.2a) can be found in PYI].

It is to be noted that *all* the dependence on b_1 is contained in $\tilde{\alpha}_s$, Eq. (2.1b).

A last comment concerns the renormalization scheme. We have followed Ref. [3] in renormalizing α, in the $\overline{\rm MS}$ scheme; but the mass m that appears in Eqs. (2.1) is the two-loop pole mass. That is to say, it is defined by the equation,

$$S_2^{-1}(\not{p}=m,m)=0, \tag{2.3}$$

where $S_2(\not{p},m)$ is the quark propagator to two loops. One can relate m to the $\overline{MS}$ parameter, also to two-loop accuracy, using the results of Ref. [7]:

$$\bar{m}(\bar{m}^2)=m\left\{1+\frac{C_F\alpha_s(m^2)}{\pi}+(K-2C_F)\left(\frac{\alpha_s}{\pi}\right)^2\right\}^{-1},$$
$$K(n_f=4)\simeq 12.5,\quad K(n_f=3)\simeq 13.0. \tag{2.4}$$

Nonperturbative corrections are not included in Eqs. (2.1); they will be incorporated later.

III. ENERGY SHIFTS, ORDER α_s^4, Λ^4/m^4

Taking into account the expression for the Hamiltonian, Eq. (2.1), we write

$$E_{nl}=2m-m\frac{C_F^2\bar{\alpha}_s^2}{4n^2}+\sum_V\ \delta_V^{(1)}E_{nl}+\delta_{V_i^L}^{(2)}E_{nl}+\delta_{\rm NP}E_{nl}. \tag{3.1}$$

We define generally the analogue of the Bohr radius,

$$a(\mu^2)=\frac{2}{mC_F\tilde{\alpha}_s(\mu^2)}.$$

The explicit expression for the different $\delta_V^{(1)}E_{nl}$, $\delta_{V_1^{(L)}}^{(2)}E_{nl}$, and $\delta_{\rm NP}E_{nl}$ can be found in PYI, Here we just mention that the dominant nonperturbative corrections for very heavy quark-antiquark systems are associated with the gluon condensate $\langle\alpha_sG^2\rangle$. Because $\langle\alpha_sG^2\rangle\sim\Lambda^4$, $\delta_{\rm NP}E_{nl}$ is of order $(\Lambda/m)^4$. Besides the corrections included in Eq. (3.1), there are a few pieces of the higher order perturbative and nonperturbative (NP) corrections that are known; they can be found discussed in PYI and will be used in the estimation of the error.

IV. NUMERICAL RESULTS

Using formula (3.1) one evaluates the quark mass and spectrum. The formula for other properties of heavy quarkonium systems, such as the hyperfine splitting and the decay of the Υ into e^+e^-, are as in PYI. We take

$$\Lambda(n_f=4,\ \text{three loops})=0.23^{+0.08}_{-0.05}\ \text{GeV}\ [\alpha_s(M_Z^2)$$
$$\simeq 0.114^{+0.006}_{-0.004}], \tag{4.1a}$$

and for the gluon condensate, very poorly known,

$$\langle\alpha_sG^2\rangle=0.06\pm0.02\ \text{GeV}^4. \tag{4.1b}$$

This value of $\alpha_s(M_Z^2)$ is slightly smaller than, though compatible with, the world average $\alpha_s(M_Z^2)=0.118$. We

[2]For the values of the constants other than b_1 entering above formulas, cf. PYI.

have preferred our value, which is obtained by averaging measurements performed at *spacelike* momenta; see the recent review of Bethke [8].

Another matter to be discussed is the choice of the renormalization point μ. As discussed in PYI a natural value for this parameter is

$$\mu=\frac{2}{na}, \tag{4.2}$$

for states with the principal quantum number n, and this will be our choice. For states with $n=1$ the results of the calculation will turn out to depend little on the value of μ, provided it is reasonably close to Eq. (4.2). Higher states are another matter; we will discuss our choices when we consider them.

The 10 state of $\bar{b}b$ and the mass m_b. As stated, we select, for the Υ state, $\mu=2/a$. We then use Eq. (3.1) to obtain the values of the b quark mass. The results are reported below; the errors correspond to the errors in Eqs. (4.1a), (4.1b). In the estimate of the errors, the condition $\mu=2/a$ is maintained satisfied when varying Λ, while for the error due to the variation of μ the other parameters are kept fixed (i.e., one no longer has then $\mu=2/a$). The dependence of m_b on μ should be taken as an indication of the theoretical uncertainty of our calculation. With Schröder's value for b_1,

$$\begin{aligned} m_b &= 5.001^{+0.097}_{-0.061}(\Lambda) \\ &\quad \mp 0.005(\langle\alpha_s G^2\rangle)^{-0.025}_{+0.037}\ (\text{vary}\ \ \mu^2\ \ \text{by}\ \ 25\%) \\ &\quad \pm 0.006\ (\text{other th uncertainty}), \\ \bar{m}_b(\bar{m}_b^2) &= 4.454^{+0.028}_{-0.015}(\Lambda)\mp 0.005(\langle\alpha_s G^2\rangle)^{-0.024}_{+0.035} \\ &\quad (\text{vary}\ \ \mu^2\ \ \text{by}\ \ 25\%) \\ &\quad \pm 0.006\ (\text{other th uncertainty}). \end{aligned} \tag{4.3}$$

The values of $\mu^2,\ \alpha_s(\mu^2),\ \tilde{\alpha}_s(\mu^2)$ are, respectively,

$$\mu^2=6.632\quad \text{GeV}^2,\quad \alpha_s(\mu^2)=0.246,\quad \tilde{\alpha}_s(\mu^2)=0.386.$$

The piece denoted by the expression ''other th uncertainty'' in Eqs. (4.3) refers to the error coming from higher dimensional operators and higher order perturbative terms; it can be found discussed in PYI. It is comfortably smaller than the errors due to the uncertainty in Λ, $\langle\alpha_s G^2\rangle$. If we omit these errors, so as not to double count them, and consider that the theoretical error is only that due to varying μ^2 by 25%, and compose all the errors quadratically, then we obtain the estimates reported in the Introduction, Eqs. (1.1).

$M(\Upsilon)-M(\eta_b)$ and the decay $\Upsilon\to e^+e^-$. The expressions for the hyperfine splitting and the decay of the Υ into e^+e^- are as in PYI. They depend on b_1 only indirectly, through the preferred values of m, μ. We have the numerical results, using the Schröder calculation

$$M(\Upsilon)-\mathrm{M}(\eta)=46.6^{+10.9}_{-3.5}(\Lambda)^{+5.5}_{-5.2}(\langle\alpha_s G^2\rangle)^{+8.3}_{-11.1} \quad (\mu^2=6.632\pm 25\,\%) \tag{4.4}$$

and

$$\Gamma(\Upsilon\to e^+e^-)=1.07^{+0.11}_{+0.01}(\Lambda)^{+0.12}_{-0.11}(\langle\alpha_s G^2\rangle)^{+0.21}_{-0.26} \times(\mu^2=6.632\pm 25\,\%) \tag{4.5}$$

practically unchanged from PYI. Note that, when varying Λ, $\langle\alpha_s G^2\rangle$, we have varied m_b according to Eq. (4.3), but we have not varied m_b when varying μ. Note also that the corrections are here fairly large, in particular, as a result of the large size of the radiative correction to the decay [9]. Composing the errors as for the b quark we find the results reported in the Introduction.

Higher order NP corrections due to the higher dimensional operators are also known for the decay rate (see PYI). Size corrections, however, are not known now.

The result for the decay is in reasonable agreement with experiment,

$$\Gamma_{\text{expt}}(\Upsilon\to e^+e^-)=1.320\pm 0.04\ \ \text{keV}.$$

Higher states $(n=2)$ of $\bar{b}b$. From Eq. (4.2) a natural choice of scale is now $\mu=1/\alpha=2.860\,\text{GeV}^2$. If we take this, adding or subtracting a 25% to estimate the dependence of the calculation on the choice of scale, then we obtain the results, with the Schröder value of b_1,

$$\begin{aligned} M(20,\text{th})-M(20,\text{expt}) &= 363^{+310}_{-324}\ \ \text{MeV} \\ &\quad (\mu^2=2.86\pm 25\,\%), \\ M(21,\text{th})-M(21,\text{expt}) &= 208^{+205}_{-216}\ \ \text{MeV} \\ &\quad (\mu^2=2.86\pm 25\,\%). \end{aligned} \tag{4.6}$$

The 10 state of $\bar{c}c$ and the mass m_c. The value of the parameter Λ used now, corresponding to that in Eq. (4.1a), is

$$\Lambda(n_f=3,\text{three loops})=0.30^{+0.09}_{-0.05}\ \ \text{GeV}.$$

The values for the c-quark mass, deduced from the J/ψ mass, are now

$$\begin{aligned} m_c &= 1.866^{+0.154}_{-0.091}(\Lambda)\mp 0.014(\langle\alpha_s G^2\rangle)^{-0.096}_{+0.149} \\ &\quad (\text{varying}\ \ \mu^2\ \ \text{by}\ \ 25\%) \\ &\quad \pm 0.014\ (\text{th uncertainty}), \\ \bar{m}_c(\bar{m}_c^2) &= 1.542^{+0.085}_{-0.053}(\Lambda)\mp 0.013(\langle\alpha_s G^2\rangle)^{-0.089}_{+0.138} \\ &\quad (\text{varying}\ \ \mu^2\ \ \text{by}\ \ 25\%) \\ &\quad \pm 0.013\ (\text{th uncertainty})\quad (S), \end{aligned} \tag{4.7}$$

and $\mu^2=2.460\,\text{GeV}^2$ now. Composing the errors as for the b quark, we find the results reported in the Introduction, Eqs. (1.1).

V. DISCUSSION

The discussion of PYI holds valid for the calculations using both Peter's and Schröder's evaluations, with one point of difference. If we believe Schröder's value of b_1, then the two-loop corrections are smaller. For example, with Peter's value we had

$$a_1\alpha_s/\pi \simeq 0.11, \quad b_1\alpha_s^2/\pi^2 \simeq 0.14, \quad (P),$$

while with Schröder's the first is almost unchanged, but the second becomes $b_1\alpha_s^2/\pi^2 \simeq 0.081 \quad (S)$.

We now would like to briefly reread our results within an effective field theory framework (for example, see Ref. [10]). This formalism takes advantage of the existence of three widely separated scales in heavy quarkonium systems: m, $m\alpha_s$, and $m\alpha_s^2$. In this work, we have considered that one can deal with the two first scales in a perturbative fashion [$\alpha_s(m) \ll 1$ and $\alpha_s(m\alpha_s) \ll 1$]. The rigorous procedure to be certain about these assumptions is by checking the convergence of the corrections. Moreover, one can also see that typical values of m and $m\alpha_s$ are much larger than Λ, especially for the $n=1$ bottomonium state. We still have to address the question of how to deal with effects at the scale $m\alpha_s^2$, usually named retardation effects. One could wonder whether to still work perturbatively at this scale. We do not face this problem here since these eventual perturbative effects first appear at $O(m\alpha_s^5)$, beyond the accuracy aimed at in this work. The leading nonperturbative effects, which appear at scales of $O(m\alpha_s^2)$ or smaller, have been taken into account by considering the correction to the energy due to local condensates, $\delta_{\mathrm{NP}}E_{nl}$. Depending on nonperturbative constants, the relative importance of this correction cannot really be estimated until its explicit computation (although, obviously, $\delta_{\mathrm{NP}}E_{nl}$ has to be smaller than the Coulomb binding energy for consistency). The use of local condensates relies on the assumption that nonperturbative corrections due to higher dimensional condensates that are parametrically smaller by a factor of $\Lambda^2/m^2(C_F\alpha_s)^4$ are indeed subleading. We have explicitly checked this assumption by computing the next-to-leading nonperturbative corrections and comparing them with respect to the leading ones. It is true, however, that these higher order corrections depend on poorly known condensates.

Let us finally note again that, although the pole mass m is correct to $O(\alpha_s^4)$, the $\overline{\mathrm{MS}}$ parameter $\overline{m}$ is only correct to $O(\alpha_s^2)$.

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